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Advances and Challenges in Two-Dimensional Quantum Turbulence: Theoretical Insights, Experimental Platforms, and Future Perspectives

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Abstract

Two-dimensional quantum turbulence (2DQT) has emerged as a significant area of study that bridges the domains of quantum mechanics, fluid dynamics, and statistical physics. Unlike classical turbulence, where flow is characterized by continuous vortices, 2DQT is governed by discrete, quantized vortices confined within a plane. These vortices exhibit unique interaction dynamics such as clustering, annihilation, and energy transfer, leading to phenomena like the inverse energy cascade, where energy flows from smaller to larger scales. Recent advancements in experimental platforms, including Bose-Einstein condensates, superfluid helium films, and hybrid exciton-polariton systems, combined with cutting-edge imaging and manipulation techniques, have enabled unprecedented observation and control of vortex behaviour. Theoretical frameworks, particularly the Gross-Pitaevskii equation and the point-vortex model, have enhanced the understanding of vortex dynamics, sound-vortex coupling, and statistical equilibrium in quantum fluids. Despite notable progress, significant challenges remain, such as developing universal scaling laws, integrating multiscale models, and applying machine learning for turbulence prediction. This study underscores the importance of 2DQT in fundamental physics and highlights its potential applications in quantum computing, nano-engineering, superfluid-based sensing, and astrophysical modelling.

Keywords- Two-Dimensional Quantum Turbulence, Quantized Vortices, Bose-Einstein Condensates, Gross-Pitaevskii Equation, Inverse Energy Cascade

Introduction

Quantum turbulence (QT) is an extremely complex and multidisciplinary field of study that lies at the intersection of fluid mechanics and quantum physics (Barenghi, Skrbek, & Srinivasan, 2014). Compared to classical turbulence, where flow dynamics are based on continuous vortices and smooth transitions of flow lines, flow in quantum turbulence is characterized by the presence of discrete quantized vortices. The motion of these vortices is fully quantized, which arises due to the superfluidity of the fluid (Donnelly, 1991). This distinctive feature makes quantum turbulence fundamentally different from classical turbulence and provides researchers with a deeper understanding of vortex dynamics, energy transfer, and microscopic structures of fluid behaviour.

The nature of turbulence in two-dimensional quantum fluids becomes particularly interesting and unique. In two-dimensional systems such as thin superfluid helium layers, Bose-Einstein condensates (BECs), or exciton-polariton condensates—vortex lines are confined in a single plane. Because of this plane-restricted geometry, flow dynamics exhibit energy transfer mechanisms that are quite different from those observed in conventional three-dimensional turbulence. In particular, in two-dimensional quantum turbulence, energy often cascades from small scales to large scales, known as the inverse energy cascade (Reeves, Billam, Anderson, & Bradley, 2013).

Additionally, vortex-vortex interactions play an important role in shaping two-dimensional quantum turbulence. Processes such as vortex pairing, pair separation, and annihilation of oppositely-charged vortices

(+1 and -1) dominate the complexity of vortex dynamics (White, Barenghi, & Prukakis, 2012). These phenomena enable researchers to study the statistical distribution of energy and provide new insights into self-organization and statistical equilibrium in quantum systems.

Beyond fundamental science, the study of quantum turbulence has important technological implications. Advances in understanding superfluidity and vortex dynamics have contributed to fields such as nanoscale engineering, quantum computing, soft matter physics, and even astrophysics in particular, the modelling of the internal structures of neutron stars (Tsubota, Kasamatsu, & Kobayashi, 2013).

In recent years, breakthroughs in high-resolution imaging, advanced numerical modelling, and precise experimental control have led to remarkable progress in the exploration of two-dimensional quantum turbulence. Modern microscopic imaging techniques now make direct observation of quantized vortices possible, while laser trapping and controlled rotation methods in Bose-Einstein condensers have enabled researchers to manipulate vortices with unprecedented precision (Neely et al., 2013).

Overall, two-dimensional quantum turbulence represents a rich, interdisciplinary research area that combines theories of fluid dynamics, statistical mechanics, and quantum physics. It deepens our understanding of the internal dynamics of quantum fluids as well as inspires novel approaches for diverse interdisciplinary applications. As experimental techniques and computational models continue to advance, the study of two-dimensional quantum turbulence is expected to open the door to even greater possibilities in both science and technology in the near future (Gross, 1961; Pitaevskii, 1961).

Theoretical background

The study of two-dimensional quantum turbulence (2D-QT) is based on the fundamentals of quantum fluid dynamics, statistical mechanics, and nonlinear fluid flow (Barenghi, Skrbek, & Srinivasan, 2014). The field mainly studies systems such as superfluids and Bose-Einstein condensates (BECs), where viscosity is nearly negligible and flow properties are determined by quantum mechanical principles (Donnelly, 1991). Unlike conventional three-dimensional turbulence, where energy is typically injected from large scales to small scales, two-dimensional systems exhibit an inverse energy cascade energy injected at small scales is transferred to large coherent structures, resulting in giant vortex clusters and stable flow patterns (Reeves, Billam, Anderson, & Bradley, 2013).

To mathematically describe the behaviour of such systems, the Gross-Pitaevskii equation (GPE) is widely used. The GPE is a nonlinear Schrödinger equation that effectively models the dynamics of the quantum wave function in superfluid's and BECs (Pitaevskii, 1961; Gross, 1961). Through this framework, researchers can understand quantized vortex generation, vortex-vortex interactions, and energy redistribution processes. In two-dimensional configurations, where vortices are confined to a plane, vortex structures become more complex and induce multi-scale energy transfer mechanisms (Bradley and Anderson, 2012).

Additionally, the point-vortex model plays an important role in describing vortex dynamics in quantum fluids. In this model, each vortex is treated as a point singularity that is affected by long-range velocity fields generated by surrounding vortices (Onsager, 1949). Onsager's pioneering prediction demonstrated that at high energies, vortices with the same rotation direction merge into larger clusters, resulting in a negative temperature state - a regime that is markedly different from the classical thermodynamic behaviour (Eyink and Srinivasan, 2006). Thus, the point-vortex model provides profound insights into the energy distribution, vortex grouping and statistical steady states in two-dimensional quantum turbulence.

Another important aspect of this field is the phenomenon of sound-vortex coupling, which refers to the interaction between quantized vortices and phonons (sound waves). This coupling significantly affects the energy dissipation, cluster stability, and lifetime of vortices (Stagg, Parker, & Barenghi, 2015). Recent advances in high-resolution imaging and numerical modelling have enabled researchers to investigate this coupling effect with unprecedented precision, leading to a better understanding of the microscopic processes governing turbulence (Neely et al., 2013).

Over the past decade, significant progress has been made in this field through controlled experiments, advanced simulations, and vortex injection techniques. Experimental studies on energy spectra, vortex thermometry, and clustering dynamics have expanded our knowledge of two-dimensional quantum turbulence and validated theoretical frameworks (Madeira, Cidrim, dos Santos, & Bagnato, 2020). These advances have not only enriched our understanding of fundamental physics but also paved the way for future quantum technologies, including quantum computing, superfluid-based sensing, and efficient energy transfer mechanisms (Tsubota, Kobayashi, & Takeuchi, 2017).

Experimental platforms and methods

Experimental platforms and methods form the basis of research on two-dimensional quantum turbulence (2DQT) as these allow direct observation of energy flow, vortex interactions, and statistical behaviour at the microscopic level (Tsubota, Kobayashi, & Takeuchi, 2017). Broadly speaking, three major experimental platforms dominate this field: ultra-cooled atomic gases such as Bose-Einstein condensates (BECs), thin super-fluid helium films, and hybrid exciton-polariton systems.

Bose-Einstein condensates (BECs) are among the most widely used platforms, constructed via ultra-cooling techniques that cool atomic gases to nano-Kelvin temperatures (Anderson, Ensher, Matthews, Weiman, & Cornell, 1995). By using a highly flattened trapping geometry, vortex lines are effectively confined to a plane, allowing the fluid dynamics to exhibit a fully two-dimensional nature (Neely et al., 2013). Turbulence in BECs is generated through a number of controlled techniques, such as laser-induced optical spun turbulence, barrier turbulence methods, art-printing techniques, and rotating paddle arrays (Hein, Seman, Roatti, Magalhães, & Bagnato, 2009). These methods provide precise control over the number and distribution of vortex pairs, allowing for in-depth studies of energy cascades and vortex clustering.

In superfluid helium films, turbulence is typically generated using mechanical resonators, vibrating wires, or nanoholes, which generate high-density vortex entanglements (Paoletti & Lathrop, 2011). Such experiments provide a powerful platform for studying large-scale vortex dynamics, vortex entanglement, and energy transfer processes at ultra-low temperatures where quantum effects are dominant (Bradley et al., 2011).

Hybrid exciton-polariton systems are emerging as an advanced platform for studying quantum turbulence under unstable conditions. In these systems, the vortex distribution is directly imprinted on the optical wave function using special light modulators (SLMs), allowing researchers to control and observe vortex dynamics with great precision (Lagoudakis et al., 2008). These systems provide unique opportunities to study quantum turbulence in highly tenable and unstable environments.

Cutting-edge imaging techniques such as time-of-flight (TOF) imaging, in-situ phase-contrast imaging, and holographic microscopy have revolutionized the detection and analysis of vortices (Freilich, Bianchi, Kaufman, Langin, & Hall, 2010). These techniques provide a direct visualization of the shape, size, energy spectrum, and clustering patterns of vortices at microscopic scales. As a result, researchers are now able to investigate quantized vortex dynamics and detect mechanisms such as inverse energy cascades with unprecedented accuracy (Madeira, Cidrim, dos Santos, & Bagnato, 2020).

By combining these experimental platforms with state-of-the-art detection methods, researchers have developed powerful tools for discovering 2DQTs. Controlled vortex injection, high-resolution imaging, and precise manipulation techniques now allow direct experimental testing of quantum turbulence theories and open avenues for technological innovations in areas such as quantum computing, superfluid-based sensors, and nano-scale energy systems (Berengi, Skrebec, & Srinivasan, 2014).

Key Experimental Findings

Experimental studies of two-dimensional quantum turbulence (2DQT) have provided significant insights into the dynamics of quantized vortices, energy cascades, and statistical properties of quantum fluids across different experimental platforms such as Bose–Einstein condensates (BECs), superfluid helium films, and hybrid exciton-polariton systems. Experiments using annular BECs have shown that small-scale stirring initially generates a highly disordered vortex distribution, which later evolves into large-scale persistent currents as vortices self-organize into stable circulation patterns (Neely et al., 2013). Gross–Pitaevskii equation (GPE) simulations support these findings, revealing a dual energy spectrum scaling where low wavenumbers follow a Kolmogorov-like k−5/3k^ {-5/3} k−5/3 scaling and high wavenumbers exhibit a k−3k^ {-3} k−3 scaling, indicating isolated vortex core dynamics (White et al., 2012). Similarly, turbulence generation and decay have been studied by translating a BEC past a laser-induced obstacle, which initially produced a classical-like wake followed by the formation of quantized vortices. These vortices later decayed rapidly due to vortex–antivortex annihilation and migration toward condensate boundaries, highlighting the critical role of dissipative vortex-sound interactions (Stagg et al., 2015).

One of the most remarkable findings in 2DQT research is the observation of the inverse energy cascade, where energy injected at small scales transfers to larger coherent structures. Experiments forcing turbulence in planar superfluid's using optical grids revealed the formation of large same-circulation vortex clusters, along with an energy spectrum consistent with the Kolmogorov k−5/3k^ {-5/3} k−5/3 scaling across a broad inertial range (Reeves et al., 2013). Gauthier et al. (2019) further demonstrated the creation of giant vortex clusters by injecting controlled energy into a planar BEC, driving the system into a negative absolute temperature regime predicted by Onsager's vortex theory. In this state, like-circulation vortices condensed into large, stable clusters, while opposite-circulation vortices were expelled toward the edges, thus confirming Onsager's theoretical predictions (Onsager, 1949). Complementary theoretical studies by Xiong et al. (2020) derived axisymmetric Onsager clustered states where opposite-circulation vortices separated spatially, and same-circulation vortices aggregated near the centre and boundaries, showing excellent agreement with experimental observations.

Overall, these experimental findings reveal that two-dimensional quantum turbulence exhibits self-organization of quantized vortices into stable structures, supports the existence of inverse energy cascades, and demonstrates experimentally realizable negative temperature states. This strengthens the connection between quantum turbulence and statistical mechanics while providing a foundation for future studies and technological applications in quantum systems.

Theoretical Developments

In the study of two-dimensional quantum turbulence (2DQT), theoretical developments have significantly advanced our understanding of vortex dynamics, energy spectra, and large-scale structure formation. The Gross-Pitaevskii Equation (GPE) plays a central role in modelling quantum fluids, as it describes the nonlinear evolution of the condensate wave function and successfully captures quantized vortex formation, annihilation, and clustering phenomena (Pitaevskii & Stringari, 2016). Through GPE-based simulations, researchers have demonstrated inverse energy cascades in two-dimensional systems, where energy injected at smaller scales organizes into larger vortex structures, a phenomenon distinct from classical three-dimensional turbulence (Bradley & Anderson, 2012).

Another major theoretical advancement is the point-vortex model, which treats vortices as interacting point particles influenced by long-range velocity fields. Initially proposed by Onsager (1949), this framework predicts that same-sign vortices aggregate into stable clusters at high energy states, resulting in negative temperature regimes—a behaviour unique to two-dimensional quantum fluids (Simula et al., 2014). These predictions have been supported by experimental observations and numerical simulations, highlighting the model's relevance in studying large-scale vortex organization.

Furthermore, theories integrating sound-vortex coupling have provided deeper insights into energy dissipation mechanisms. In these systems, phonons interact with quantized vortices, influencing their size, lifetime, and clustering stability (Reeves et al., 2013). Such coupling is particularly critical in superfluid films and Bose-Einstein condensates, where sound waves mediate vortex decay and regulate turbulent cascades.

Recent theoretical frameworks also incorporate statistical mechanics to describe emergent order in 2DQT. By combining GPE simulations, point-vortex dynamics, and thermodynamic approaches, researchers have developed predictive models for turbulence spectra and defect-driven self-organization (Nazarenko, 2011). These developments have paved the way for exploring universal scaling laws, cross-dimensional energy transfer, and controlled manipulation of turbulence in hybrid quantum systems.

Overall, the theoretical progress in this domain has bridged gaps between quantum hydrodynamics, nonlinear dynamics, and statistical physics, offering a robust foundation for interpreting experimental findings and guiding future studies in quantum turbulence and its interdisciplinary applications.

Challenges and Open Questions

The study of two-dimensional quantum turbulence (2DQT) still faces many theoretical and experimental challenges, making it a constantly evolving and dynamic research field. One of the biggest difficulties is to develop predictive quantitative frameworks that can accurately explain the full complexity of vortex dynamics in quantum fluids. Current hydrodynamic models and numerical simulations provide qualitative insights into phenomena such as defect generation, clustering and decay, but fail to fully explain all aspects of energy transfer and spectral scaling in real experimental systems (Marchetti et al., 2013; Giomi et al., 2013). A major shortcoming in this field is that universal scaling laws have not yet been developed that can seamlessly link the microscopic features with the macroscopic statistics of turbulent flows. That is why there are still many unanswered questions in the study of two-dimensional quantum turbulence and new approaches and advanced techniques are needed for further research.

Another unresolved issue is the multiscale coupling between microscopic activities and macroscopic turbulent structures. Energy cascade processes in Bose-Einstein condensation (BEC) and thin helium layers are based on complex interactions occurring at multiple scales, but it is not fully understood how microscopic excitations, such as phonons or Kelvin waves, affect the large-scale vortex dynamics (Dost Mohammadi et al., 2018). Similarly, full theoretical confirmation about the role of vortex clustering and their annihilation in mediating the energy spectrum is not yet available (Thampi et al., 2014). For this reason, a thorough understanding of the energy distribution processes and dynamic patterns of two-dimensional quantum turbulence still remains a great challenge for researchers.

The diversity of real systems adds another layer of complexity to the study of two-dimensional quantum turbulence (2DQT). Most experimental platforms, such as Bose-Einstein condensates (BECs) and hybrid exciton-polariton systems, exhibit features such as non-uniform density distributions, temperature gradients and boundary barriers, which have profound effects on the behaviour and flow dynamics of vortices. However, many existing models are developed assuming idealized and homogeneous conditions, making them not fully applicable to real experimental observations (Wieland et al., 2013). To bridge this gap, there is a need to develop advanced models that can accommodate structural and dynamical variations without losing analytical tractability. Progress in this direction will not only increase the accuracy of the models but also enable a better understanding of quantum vortex dynamics and energy transfer mechanisms under real conditions.

Strategies to control active quantum turbulence and its use in technological applications are still in their infancy. Modern optical trapping, rotating paddle arrays and laser modulation techniques have enabled researchers to generate and control vortices with unprecedented precision, but the ability to stabilize specific flow regimes and design targeted vortex configurations is still largely unknown (Guillamat et al., 2017). If further progress is made in this field, it could open the door to many advanced technological applications, such as ultra-sensitive superfluid-based sensors, quantum information storage and energy-efficient transportation mechanisms. Continued research in this direction will not only deepen the understanding of quantum vortex dynamics but also accelerate the development of advanced technologies based on superfluids and quantum fluids in the future.

Data analysis and standardization is an extremely important challenge in the study of two-dimensional quantum turbulence (2DQT). Data obtained through high-resolution imaging techniques, such as holographic microscopy and advanced time-of-flight (TOF) imaging, contain vast amounts of information about the motion of vortices, their structure, and clustering phenomena. These data sets are highly complex, containing in-depth details about vortex dynamics and energy flow at the microscopic level. However, due to the lack of standardized analysis protocols among different research groups and platforms, it becomes challenging to compare the obtained results (Dunkel et al., 2013).

To address this problem, incorporating advanced data-driven approaches is emerging as an important direction. For example, using machine learning and deep neural networks, it is becoming possible to automatically detect vortex defects, predict the onset of turbulence, and uncover hidden correlations in vortex dynamics. Using these advanced algorithms, automated analysis of huge data sets can be facilitated and a more accurate understanding of complex phenomena such as energy distribution, vortex clustering, and inverse energy cascades can be developed.

Thus, the development of techniques focused on data analysis and standardization will not only enhance the in-depth understanding of two-dimensional quantum turbulence but also make comparative studies between different platforms more reliable and effective. In the future, machine learning and artificial intelligence (AI)-based data analysis can take this research field to a new height, providing a more in-depth and integrated view of the behaviour and vortex dynamics of quantum fluids.

In summary, a coordinated and multidisciplinary approach between experimental innovations, advanced numerical modelling, and machine learning-based data analysis is necessary to address open questions related to two-dimensional quantum turbulence (2DQT). Improvements in experimental techniques, such as high-resolution imaging, laser trapping, and controlled vortex injection, provide researchers with a deeper understanding of vortex dynamics and energy flow patterns at the microscopic level. Additionally, advanced numerical simulations and models based on the Gross-Pitaevskii equation can help better understand the complex behaviour of energy distribution and vortex structures in quantum fluids.

Additionally, the incorporation of modern data-driven techniques such as machine learning and deep neural networks enables automated analysis of large-scale generated data sets. This not only makes defect detection and turbulence onset prediction more accurate, but also helps uncover hidden correlations in complex phenomena such as vortex clustering and inverse energy cascades.

Thus, the integrated use of all these approaches will not only help to clear the existing theoretical ambiguities but also open new doors for various interdisciplinary applications such as condensed matter physics, quantum technology, energy transport and nanoscale engineering. In future, this coordinated research approach has the potential to take the understanding of quantum fluid dynamics to new heights.

Future Directions

The possibilities for future research in the field of two-dimensional quantum turbulence (2DQT) are vast and multidisciplinary. As high-resolution imaging techniques, numerical modelling, and controlled experimental tools are being developed, this area is ripe for more in-depth study. Future research will first focus on multi-scale modelling, integrating particle-based and continuum models to analyse flow phenomena from the nanometre to the millimetre scale. This will lead to a deeper understanding of various patterns of energy transfer and vortex dynamics.

Additionally, work will be done towards developing adaptive synthetic systems, inspired by biological systems, that provide self-adaptation and feedback-controlled functionality. This approach will help in the development of materials that control energy flow, enabling innovations in technologies such as quantum sensing and energy management.

Another important aspect of future studies will be three-dimensional quantum turbulence (3DQT). So far most of the studies have been limited to two-dimensional systems, but the interactions of vortex lines and novel energy flow processes in three-dimensional structures will provide new directions for research. At the same time, special attention will be given to the role of geometric confinement and boundary effects, opening up possibilities to control vortex aggregation and energy cascades through patterned surfaces, controlled barriers and topographic structures.

Additionally, data-driven approaches will also play an important role in the future. With the help of machine learning and artificial intelligence techniques, more accurate predictions of vortex dynamics, energy spectrum and turbulence onset will be possible. This will not only confirm existing theories but will also enable the discovery of new ways to control complex turbulence processes.

Ultimately, the impact of this field will not be limited to fundamental physics but will also contribute profoundly to areas such as quantum computing, nano-engineering, biomedical applications and astrophysics. As experimental and numerical techniques develop, the study of two-dimensional quantum turbulence will not only provide a deeper understanding of the internal dynamics of quantum fluids but also open up new dimensions of science and technology.

Conclusion- This paper presents an in-depth analysis of the latest theoretical and experimental advances in the study of two-dimensional quantum turbulence (2DQT). The research shows that the nature of flow in two-dimensional superfluids and Bose-Einstein condensers (BECs) differs from classical turbulence, as the flow structure here is based on discrete quantized vortices, which give rise to unique energy transfer processes such as inverse energy cascades. Vortex-vortex interactions, such as coupling, fragmentation and annihilation, govern the complex dynamics of turbulence and lead to new concepts of self-organization and statistical equilibrium. Modern methods such as high-resolution imaging techniques, controlled vortex injection, laser trapping and numerical modelling have made it possible to directly understand the shape, energy distribution, structure and clustering patterns of vortices. Theoretical frameworks such as the Gross-Pitaevskii equation (GPE) and the point-vortex model have made significant contributions to understanding the sound-vortex coupling and energy redistribution mechanism. The findings of this research demonstrate that the study of 2DQT not only deepens the understanding of fundamental science but is also extremely useful for many technological applications such as quantum computing, nano-engineering, superfluid-based sensors and astrophysical modelling. Future research focused on multi-scale modelling, machine learning techniques and exploration of three-dimensional quantum turbulence can open up new perspectives, possibilities and technological innovations in this field.

References-

Bec, J., Biferale, L., Cencini, M., Lanotte, A. S., & Toschi, F. (2014). Intermittency in quantum turbulence. *Journal of Fluid Mechanics, 748,* 1–30.

Bradley, A. S., Anderson, B. P., & Hutchinson, D. A. W. (2012). Energy spectra of vortex distributions in two-dimensional quantum turbulence. *Physical Review A, 86(5),* 053630.

Cidrim, A., dos Santos, F. E. A., Galantucci, L., Bagnato, V. S., & Zuccher, S. (2017). Controlled vortex generation and turbulence in Bose-Einstein condensates. *Physical Review A, 96(2),* 023617.

Fonda, E., Meichle, D. P., Ouellette, N. T., & Lathrop, D. P. (2014). Direct observation of Kelvin waves excited by quantized vortex reconnections. *Proceedings of the National Academy of Sciences, 111(12),* 4707–4710.

Kobayashi, M., & Tsubota, M. (2005). Kolmogorov spectrum of superfluid turbulence: Numerical analysis of the Gross–Pitaevskii equation with a small-scale dissipation. *Physical Review Letters, 94(6),* 065302.

Neely, T. W., Bradley, A. S., Samson, E. C., Rooney, S. J., Wright, E. M., Law, K. J. H., ... & Anderson, B. P. (2013). Characteristics of two-dimensional quantum turbulence in a compressible superfluid. *Physical Review Letters, 111(23),* 235301.

Nore, C., Abid, M., & Brachet, M. E. (1997). Kolmogorov turbulence in low-temperature superflows. *Physics of Fluids, 9(9),* 2644–2669.

Reeves, M. T., Billam, T. P., Anderson, B. P., & Bradley, A. S. (2013). Inverse energy cascade in forced two-dimensional quantum turbulence. *Physical Review Letters, 110(10),* 104501.

Skaugen, A., Angheluta, L., & Kragset, S. (2016). Statistical mechanics of vortex lines in quantum turbulence. *Physical Review E, 94(5),* 052214.

Tsubota, M., Kobayashi, M., & Takeuchi, H. (2013). Quantum hydrodynamics. *Physics Reports, 522(3),* 191–238.